The Hitchin fibration II

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G reductive gp $/ k = \bar{k}$

W Weyl gp, char $k \nmid |W|$

$$\mathfrak{g} = \mathrm{Lie}(G)$$
$$\cup$$
$$\mathfrak{t} = \mathrm{Lie}(T)$$

Chevalley map:

$$\begin{tikzcd} \mathfrak{g} \arrow[d] \arrow[dr] & \\ {[\mathfrak{g}/G]} \arrow[r] & \mathfrak{c} := \mathfrak{t}/W \end{tikzcd}$$

$X \xrightarrow{f} \operatorname{Spec} k$

smooth proj. irred. curve

$\mathcal{L} \in \mathrm{Pic}(X)$

$\deg \mathcal{L} > 2g_X$ or $\mathcal{L} \cong \omega_X$

$\mathfrak{g}_{\mathcal{L}} := \mathfrak{g} \otimes \mathcal{L}$

$\mathfrak{c}_{\mathcal{L}} := \mathfrak{c} \otimes \mathcal{L}$

Hitchin map:

$$\begin{tikzcd} f_*[\mathfrak{g}_{\mathcal{L}}/G] \arrow[r, "h"] \arrow[d, equal] & f_* \mathfrak{c}_{\mathcal{L}} \arrow[d, equal] \\ \mathcal{H}iggs & \mathcal{B} \end{tikzcd}$$

1. The torsor $\mathcal{H}iggs^{reg}$

Recall: $\exists$ comm. gp. scheme $\mathcal{J}_{\mathcal{L}}$ on $\mathfrak{c}_{\mathcal{L}}$ (descent of universal centralizer $I_{\mathcal{L}}$ on $\mathfrak{g}^{reg}_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$)

sth

$$[\mathfrak{g}^{reg}_{\mathcal{L}} / G] \longrightarrow \mathfrak{c}_{\mathcal{L}}$$

is a $\mathcal{J}_{\mathcal{L}}$-gerbe.

Def The Picard stack $\mathcal{P}$ of $\mathcal{J}$-torsors is given by

$$(S \xrightarrow{b} B) \longmapsto \langle b^*\mathcal{J}\text{-torsors on } X \times S \rangle$$

(view b as a morphism $b : X \times S \to \mathfrak{c}_{\mathcal{L}}$).

Thm $\mathcal{H}iggs^{reg} := f_*[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ is a $\mathcal{P}$-torsor over B.

Proof.
① Action $\mathcal{P} \times_B \mathcal{H}iggs^{reg} \xrightarrow{\alpha} \mathcal{H}iggs^{reg}$:

Let $\mathcal{M} \in \mathcal{P}(S)$ over $b \in B(S)$.
$(E, \theta) \in \mathcal{H}iggs^{reg}(S)$

gerbe $\rightsquigarrow$ $b^*(\mathcal{J}_{\mathcal{L}}) \xrightarrow{\sim} Aut(E, \theta)$

$\rightsquigarrow$ put $\alpha(\mathcal{M}, (E, \theta)) := (E, \theta) \overset{b^*(\mathcal{J}_{\mathcal{L}})}{\times} \mathcal{M}$

② Torsor property:
For fixed $(E, \theta) \in \mathcal{H}iggs^{reg}(S)$ over $b \in B(S)$,
the functor $\mathcal{P}_b \xrightarrow{\sim} \mathcal{H}iggs^{reg}_b = h^{-1}(b)$

$\mathcal{M} \longmapsto \alpha(\mathcal{M}, (E, \theta)) = (E', \theta')$

is an equivalence with inverse $(E', \theta') \mapsto \mathcal{H}om((E, \theta), (E', \theta'))$
(use gerbe property)

$\square$

Rem. If $\exists\, \mathcal{L}^{1/2} \in \mathrm{Pic}(X)$ with $(\mathcal{L}^{1/2})^{\otimes 2} \simeq \mathcal{L}$, then $\mathcal{H}iggs^{rg} \to \mathcal{B}$ admits a section:

$$
\begin{tikzcd}
\mathcal{H}iggs^{rg}(S) = \left\langle \begin{array}{l} X \times S \xrightarrow{f} \left[\mathfrak{g}^{rg} / G \times \mathbb{G}_m \right] \\ \text{over } X \times S \xrightarrow{[p_i^* \mathcal{L}]} B\mathbb{G}_m \end{array} \right\rangle \arrow[d, "h"] \\
\mathcal{B}(S) = \left\langle \begin{array}{l} X \times S \longrightarrow [\mathfrak{c}/\mathbb{G}_m] \\ \text{over } [p_i^* \mathcal{L}] \end{array} \right\rangle \arrow[d, "\wr\|"', "\text{— choice of } \mathcal{L}^{1/2}"] \\
\mathcal{B}^{1/2}(S) := \left\langle \begin{array}{l} X \times S \longrightarrow [\mathfrak{c}/\mathbb{G}_m^{[2]}] \\ \text{over } [p_i^* \mathcal{L}^{1/2}] \end{array} \right\rangle \arrow[uu, dashed, bend left]
\end{tikzcd}
$$

via Kostant section $[\mathfrak{c}/\mathbb{G}_m^{[2]}] \longrightarrow [\mathfrak{g}^{rg}/G \times \mathbb{G}_m]$.

2. Cameral curves & the discriminant

Def. For $b \in \mathcal{B}(S)$, the cameral cover $\tilde{X}_b \to X_b = X \times S$ is defined by

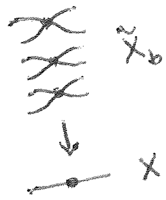
$$
\begin{tikzcd}
\tilde{X}_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{t}_{\mathcal{L}} \arrow[d] \\
X_b \arrow[r] & \mathfrak{c}_{\mathcal{L}}
\end{tikzcd}
$$

<u>Lemma.</u> $\exists$ divisor $\Delta \subset B$ sth $\forall b \in \underbrace{(B \setminus \Delta)}_{=: B^\circ}(k)$,

(a) $\mathcal{H}iggs_b^{reg} = \mathcal{H}iggs_b$

(b) $\tilde{X}_b$ is smooth

& $\tilde{X}_b \longrightarrow X$ simply branched



(c) $\mathrm{Aut}(E, \theta) = Z(G)$ $\forall (E, \theta) \in \mathcal{H}iggs_b(k)$.

Proof (sketch).

(a) Branch locus $\partial \subset c$ of $t \to c = t/W$
is defined by $\quad \mathrm{discr} := \prod_{\alpha \in \Phi} d\alpha \in k[t]^W$

Let $B^\circ(k) = \{ b \in B(k) \mid X \xrightarrow{b} c_{\mathcal{L}}$ intersects $\partial_{\mathcal{L}}$ only transversely & in its smooth part $\}$

Claim: For $b \in B^\circ(k)$, $(E, \theta) \in h^{-1}(b)$,

$$\begin{array}{ccc} X & \xrightarrow{(E,\theta)} & [\mathfrak{g}_{\mathcal{L}}/G] \\ & \exists \searrow & \cup \\ & & [\mathfrak{g}_{\mathcal{L}}^{reg}/G] \end{array}$$

Work locally on X
sth $\mathcal{L}$ & the G-torsor given by $X \to [\mathfrak{g}_{\mathcal{L}}/G]$ get trivial.

$\Rightarrow$ enough to check:

If $\varphi : \operatorname{Spec} k[\![t]\!] \to \mathfrak{g}$ & $\varphi(0) \in \mathfrak{g} \setminus \mathfrak{g}^{reg}$ (closed pt)

then $\varphi^*(\mathrm{discr}) \in m_0^2 = t^2 k[\![t]\!]$.

viewed as fct^n on $\mathfrak{g}$ via $k[\mathfrak{t}]^W \cong k[\mathfrak{g}]^G \subset k[G]$

(b) Local computation [Ngo, Lemme fond., 4.7.3]

(c) $H^0(X, b^* J_{\mathcal{L}}) = Z(G)$ $\square$

from Galois description of $J_{\mathcal{L}}$ (see below).

3. Description of $\mathcal{P}_b$ by Prym varieties

Recall: $J_{\mathcal{L}}(U) \simeq \left\{ \begin{array}{l} \tilde{U} := U \times_{\mathfrak{c}_{\mathcal{L}}} \mathfrak{t}_{\mathcal{L}} \xrightarrow{f} T \\ W\text{-equivariant sth } \alpha \circ f|_{\tilde{U}^\alpha} = 1 \\ \forall \alpha \in \Phi,\ \tilde{U}^\alpha := \operatorname{Fix}(s_\alpha) \subset \tilde{U} \end{array} \right\}$

$$\subseteq \underbrace{\pi_*(\tilde{X} \times T)^W}_{=:\, J^1_{\mathcal{L}}}(U),$$

for $\tilde{X} := \mathfrak{t}_{\mathcal{L}}$ universal cameral curve.

$$\pi^* \mathcal{J}_\mathcal{L} \longrightarrow \pi^* \mathcal{J}_\mathcal{L}^1 \longrightarrow T_{\tilde{X}}$$

$$\Big(\underset{\cap}{E} \longmapsto \underset{\cap}{\pi^* E \overset{\pi^*\mathcal{J}^1_\mathcal{L}}{\times} T} \Big)$$

gives

$$\begin{array}{ccccc} \mathcal{P}_b & \longrightarrow & \mathcal{P}_b^1 & \longrightarrow & \mathrm{Bun}_T^W(\tilde{X}_b) \\ \parallel & & \parallel & & \\ \mathrm{Tors}_{\mathcal{J}_{\mathcal{L},b}} & & \mathrm{Tors}_{\mathcal{J}^1_{\mathcal{L},b}} & & \end{array}$$

(later $b \in B^\circ(k)$)

where

$$\mathrm{Bun}_T^W(\tilde{X})(S) := \left\langle (E, \gamma = (\gamma_w)_{w \in W}) \;\middle|\; \begin{array}{l} E \ T\text{-torsor} / \tilde{X}_S, \\ \gamma_w : w(E) \xrightarrow{\sim} E \\ \text{compatible iso's} \end{array} \right\rangle$$

$(S \to B)$

Here $w(E) := (w^{-1})^*(E) \overset{T,w}{\times} T$.

<u>Rem.</u> ① Have natural map

$$\mathrm{Bun}_T^W(\tilde{X}_b) \longrightarrow \mathrm{Prym}_\Lambda(\tilde{X}_b) := \mathrm{Hom}(\Lambda, \mathrm{Pic}(\tilde{X}_b))^W$$

$$(E, \gamma) \longmapsto (\lambda \mapsto E \overset{T,\lambda}{\times} \mathbb{G}_m)$$

for $\Lambda := X^*(G) = \mathrm{Hom}(T, \mathbb{G}_m)$ weight lattice.

Here $\mathrm{Prym}_\Lambda(\tilde{X}_b)^\circ$ is an abelian variety: $\boxed{\text{if } \tilde{X}_b \text{ smooth}}$

For any $\lambda \in \Lambda \setminus \{0\}$, $ev_\lambda : \mathrm{Prym}_\Lambda(\tilde{X}_b) \to \mathrm{Pic}(\tilde{X}_b)$ has finite kernel since $[\Lambda : \mathbb{Z}[W]\cdot\lambda] < \infty$.

② On coarse moduli,

$$\mathcal{P}_b^{(1)} := \mathcal{P}_b^{(1)} /\!/ Z(G) \longrightarrow \mathrm{Prym}_\Lambda(\tilde{X}_b) \qquad (b \in B^\circ(k))$$

have finite ker & coker

$\Longrightarrow$ $\mathcal{P}_b^\circ$ abelian var. isogenous to $\mathrm{Prym}_\Lambda(\tilde{X}_b)^\circ$

& $\mathcal{P}_b$ Beilinson 1-motive.

<u>Ex.</u> $G = Gl_n$:

$$\begin{tikzcd} \tilde{X}_b \arrow[dr] \arrow[dd, "S_n"'] & \\ & Y_b := \tilde{X}_b / S_{n-1} \arrow[dl] \\ X & \end{tikzcd}$$

"cameral cover"

"spectral cover"

descent (exercise!)

$$\mathcal{P}_b \simeq \mathcal{P}ic(Y_b) \simeq \left\{ \begin{array}{l} S_{n-1}\text{-equivariant line bdles } (\mathcal{M}, \gamma) \text{ on } \tilde{X}_b \\ \text{s.th } \forall \sigma \in S_{n-1},\quad \gamma_\sigma : \sigma(\mathcal{M})|_{\tilde{X}_b^\sigma} \xrightarrow{\sim} \mathcal{M}|_{\tilde{X}_b^\sigma} \\ \qquad \text{is the tautological iso} \end{array} \right\}$$

$$\simeq \left\{ \begin{array}{l} (E, \gamma) \in \mathrm{Bun}_T^{S_n}(\tilde{X}_b) \text{ s.th} \\ \forall \sigma \in S_{n-1}, \\ \gamma_\sigma : \sigma(E) \overset{T,\lambda_n}{\times} \mathbb{G}_m|_{\tilde{X}_b^\sigma} \xrightarrow{\sim} E \overset{T,\lambda_n}{\times} \mathbb{G}_m|_{\tilde{X}_b^\sigma} \\ \qquad \text{is the tautological iso} \end{array} \right\}$$

via $\mathcal{M} = (E \overset{T,\lambda_n}{\times} \mathbb{G}_m, \gamma)$

for $\lambda_n : T = \mathbb{G}_m^n \xrightarrow[pr_n]{} \mathbb{G}_m$